



## Solution of the Generalized Interval Linear Programming Problems: Pessimistic and Optimistic Approaches

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### ABSTRACT

This paper deals with linear programming problem with interval numbers as coefficients to exhibit with uncertainty. Since, the set of common intervals is not a field, we define generalized interval numbers to produce an algebraic interval field and on this field, we propose **principle of uncertainty traverse** instead of **extension principle** which permits to define operators on intervals exactly similar to the same operators on real numbers. In addition, we apply a total order on this field to transform interval linear programming into a traditional problem. The proposed order can be extended either pessimistically or optimistically. The numerical experiments are given to demonstrate the efficiency of the proposed scheme in comparison with the previous established works. The approach in this paper can be generalized to fuzzy linear programming problems taking the fuzzy cuts into account.

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## 1 Introduction

For some considerable time, linear programming (LP) has been one of the operational research techniques, which has been widely used and got many achievements in both applications and theories [1]. However the strict requirement of LP is that the data must be well defined and precise, which is often impossible in real decision problems. The traditional way to evaluate any imprecision in the parameters of an LP model is through a post-optimization analysis, with the help of sensitivity analysis and parametric programming. However, none of these methods is suitable for an overall analysis of the effects of imprecision in parameters. Another way to handle impreciseness is to model it in stochastic programming problems according to the probability theory. A third way to cope with imprecision is to resort to the theory of interval

mathematics or fuzzy sets, which give the conceptual and theoretical framework for dealing with complexity and uncertainty [2, 3]. Interval mathematics started in 1950s and came more applicable in programming soon. They can handle uncertainty aspects according to the statistical analysis which can estimate the quantities bounds. Nowadays very researchers focus on this type of programming, see [4–6]. Also, intervals can be considered in fuzzy optimization when fuzzy cuts are followed [3, 7, 8]. Interval numbers are also important in numerical analysis, see e.g., [4, 9–12].

The basic results on interval analysis has been given by Moore [9] and Boche [13]. In the latter reference, complex interval was defined which can be represented by a rectangle or a circle in the complex plane. In this kind of numbers, an interval real part and an interval imaginary part are used or a triple consisting of the center, the lower radius and the upper radius are considered similar to a disk [14]. Then Petkovic et al. [14] defined arithmetic on complex intervals. In addition, Ramot et al. [15] introduced complex fuzzy

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numbers. In their work membership functions were defined as complex valued functions.

In parallel to these extensions, Kandel et al. [16] and Friedman et al. [17] in solving fuzzy linear systems found some contradiction solutions whose their right limits are less than their left limits. This contradiction is appeared from a gap in fuzzy numbers. Respect to the practitioners' attempt for finding appropriate solutions to every system, Friedman et al. [17] accepted these unrealistic solutions and named them as weak fuzzy solutions. Allahviranloo [18] used this terminology, too. Chang and Lee [19] mentioned to the same problem in fuzzy regression and they proposed fuzzy numbers with spreads unrestricted in sign.

We believe that this gap in fuzzy numbers is a result of similar gap in intervals. So we must fill the set of intervals with new numbers at first, which simplify the calculations. Our approach is to define arithmetic operators on interval numbers applying Ganesan and Veeramani's approach [6]. For this aim, we propose a new appearance process of negative integers and we extend the operators in complex intervals to this new interval set to find a new definition for addition, multiplication, subtraction and division. We show that the produced set with such operators is an algebraic field. For demonstrating the efficiency of this scheme, we argue linear programming on this interval field and we try to solve this problem with optimistic and pessimistic ranking functions [8, 20]. In ordering category, Moore [21] is the first author which combined the previews results on ' $<$ ' on real line and ' $\subseteq$ ' on the sets, to create two transitive order relations over intervals. Then, Ishibuchi and Tanaka [22] suggested two order relations  $\leq_{LR}$  and  $\leq_{MW}$ . However these ordering were not total order. Kundu [23] defined a fuzzy preference relationship between two interval and used it to find optimal decision. More differently Sengupta and Pal [8] suggested two index for comparing two intervals numerically. But these approaches focus on the lower and upper bound of intervals. In this paper we follow Hashemi et al.' idea [3] for minimum interval cost flow problem. This approach provides a platform to take the risk into account. We show the efficiency of this approach for interval linear programming in this paper.

The rest of this paper is organized as follows: In Section 2, we review on the classical theory of arithmetic operators on intervals and we discuss on some drawbacks. In Section 3 we introduce a generalization of intervals which produces a field of intervals. Then a total order by pessimistically and optimistically behaviors is introduced. An interval linear programming with three groups of assumptions are solved in Section 4. We also gave an application of the approach in

Section 5. Section 6 ends the paper with conclusion and future directions.

## 2 Interval Numbers

An interval number  $A = [a_L, a_R]$  is the set of all real numbers  $x$ , such that  $a_L \leq x \leq a_R$ . We denote the set of interval numbers by  $IN$ . If  $a_L = a_R$ , then  $A$  is a real number or a degenerate. Interval.  $A$  is alternatively represented as  $A = \langle a, \alpha \rangle$ , where  $a = \frac{a_L + a_R}{2}$  and  $\alpha = \frac{a_R - a_L}{2}$  are center and width of interval number  $A$ , respectively. In this paper, we use the latter notation for representation of interval numbers. An interval  $A = \langle a, \alpha \rangle$  is said to be nonnegative if  $a - \alpha \geq 0$  and nonpositive if  $a + \alpha \leq 0$ .

Traditionally, arithmetic operations on interval numbers are defined by the extension principle [4, 9]. Let  $f : \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}$  be a binary operation over real numbers. Then it can be extended to the operation over interval numbers. Let  $A$  and  $B$  be two interval numbers and  $*$   $\in \{+, -, \cdot, / \}$  be a binary operation on the set of real numbers, based on the extension principle, the binary operation  $*$  over interval numbers  $A$  and  $B$  can be defined as follows [9]:

$$A * B = \{a * b | a \in A, b \in B\}. \quad (1)$$

In the case of division it is assumed that 0 not exists in  $B$ .

Thus for  $A = \langle a, \alpha \rangle$  and  $B = \langle b, \beta \rangle$ , the extended addition, subtraction, multiplication, and division are derived as

$$\langle a, \alpha \rangle + \langle b, \beta \rangle = \langle a + b, \alpha + \beta \rangle, \quad (2)$$

$$\langle a, \alpha \rangle - \langle b, \beta \rangle = \langle a - b, \alpha + \beta \rangle, \quad (3)$$

$$\langle a, \alpha \rangle \times \langle b, \beta \rangle = \langle (d + c)/2, (d - c)/2 \rangle, \quad (4)$$

$$\langle a, \alpha \rangle / \langle b, \beta \rangle = \langle (f + e)/2, (f - e)/2 \rangle, \quad (5)$$

where

$$c = \min\{(a - \alpha)(b - \beta), (a - \alpha)(b + \beta), (a + \alpha)(b - \beta), (a + \alpha)(b + \beta)\},$$

$$d = \max\{(a - \alpha)(b - \beta), (a - \alpha)(b + \beta), (a + \alpha)(b - \beta), (a + \alpha)(b + \beta)\},$$

$$e = \min\left\{\frac{a - \alpha}{b - \beta}, \frac{a - \alpha}{b + \beta}, \frac{a + \alpha}{b - \beta}, \frac{a + \alpha}{b + \beta}\right\},$$

$$f = \max\left\{\frac{a - \alpha}{b - \beta}, \frac{a - \alpha}{b + \beta}, \frac{a + \alpha}{b - \beta}, \frac{a + \alpha}{b + \beta}\right\}.$$

If both of  $A$  and  $B$  are nonnegative or nonpositive intervals, then extended multiplication and division are simplified as



$$\langle a, \alpha \rangle \times \langle b, \beta \rangle = \langle ab + \alpha\beta, a\beta + b\alpha \rangle, \quad (6)$$

$$\langle a, \alpha \rangle / \langle b, \beta \rangle = \left\langle \frac{ab + \alpha\beta}{b^2 - \beta^2}, \frac{a\beta + b\alpha}{b^2 - \beta^2} \right\rangle. \quad (7)$$

The extended interval operations 2-5 have been used in solving interval linear system of equations e.g. in Gaussian Elimination [10] and interval interpolation [4, 12]. However these operations cannot produce interval group and field.

**Definition 1.** A group is a set  $G$  which is closed under a binary operation  $*$  (that is, for any  $x, y \in G, x * y \in G$ ) and satisfies the following properties:

- (1) Identity: There is an element  $e \in G$ , such that for every  $x \in G, e * x = x * e = x$ .
- (2) Inverse: For every  $x \in G$  there is an element  $y \in G$  such that  $x * y = y * x = e$ , where again  $e$  is the identity.
- (3) Associativity: The following identity holds for every  $x, y, z \in G: x * (y * z) = (x * y) * z$ .

**Definition 2.** A group is said to be abelian if  $x * y = y * x$  for every  $x, y \in G$ .

**Definition 3.** A field is a set  $F$  which is closed under two binary operations  $+$  and  $*$  (called addition and multiplication) such that

- (1)  $F$  is an abelian group under  $+$  and
- (2)  $F - \{0\}$  (the set  $F$  without the additive identity 0) is an abelian group under  $*$ .
- (3) For each  $x, y, z \in F$  we have:

$$x * (y + z) = x * y + x * z.$$

The identity elements under addition and multiplication operations are called zero and unit elements, respectively.

It is clear that the interval  $\langle 0, 0 \rangle$  is the zero element for  $IN$ , but there is no addition inverse for interval numbers with positive width. Therefore, the set of interval numbers under binary operation 2 is not a group. Therefore, the set  $IN$  under binary operations 2 and 4 as addition and multiplication operators is not a field. So we cannot solve system of interval equations efficiently. For example, we consider the simple example presented by Hansen [4]. Assume that  $n$  intervals  $A_i, i = 1, \dots, n$  are given and for each  $i = 1, \dots, n$  we want the sum of all except the  $i^{th}$  interval. Suppose that we first compute the sum

$$S_1 = A_2 + \dots + A_n.$$

Afterwards, we want the sum

$$S_2 = A_1 + A_3 + \dots + A_n.$$

Instead of calculation the sum  $S_2$  directly, we are going to use subtraction formula 3. Note that  $S_2 =$

$S_1 + A_1 - A_2$ . So we can compute  $S_2$  by adding  $A_1$  to  $S_1$  and then canceling  $A_2$  from the result by subtraction. But  $A_2 - A_2 = \langle a_2 - a_2, a_2 + a_2 \rangle = \langle 0, 2a_2 \rangle$  is not the (degenerate) zero interval. Therefore, we cannot cancel  $A_2$  from  $S_1$  simply by subtracting unless  $A_2$  is real numbers.

Instead of subtracting using extended interval subtraction, Hansen [4] used the special cancellation rule as follows:

$$\langle a, \alpha \rangle \setminus \langle b, \beta \rangle = \langle a - b, \alpha - \beta \rangle. \quad (8)$$

This operator is similar as Hukuhara’s difference for fuzzy numbers [24, 25].

Now let us consider the following simple interval linear equation.

$$\langle 35, 2.5 \rangle + \langle x, y \rangle = \langle 50, 3.5 \rangle. \quad (9)$$

In order to solve Equation 9, we may write the following relation

$$\langle x, y \rangle = \langle 50, 3.5 \rangle - \langle 35, 2.5 \rangle = \langle 15, 6 \rangle,$$

while this solution don’t satisfy in the Equation 9. But if instead of extended subtraction we use the cancellation rule 8, then we obtain the desired solution  $\langle 15, 1 \rangle$ . Therefore, cancellation is more applicable in these cases in comparison with extended interval subtraction. Now consider the interval numbers  $A = \langle 150, 10 \rangle$  and  $B = \langle 125, 5 \rangle$ . Using cancellation rule 8 to compute  $A \setminus B$ , we obtain  $\langle 25, -5 \rangle$ . Note that  $\langle 25, -5 \rangle$  is not an interval number since the width of interval numbers must be nonnegative. So cancellation rule is not

### 3 Generalized Interval Numbers

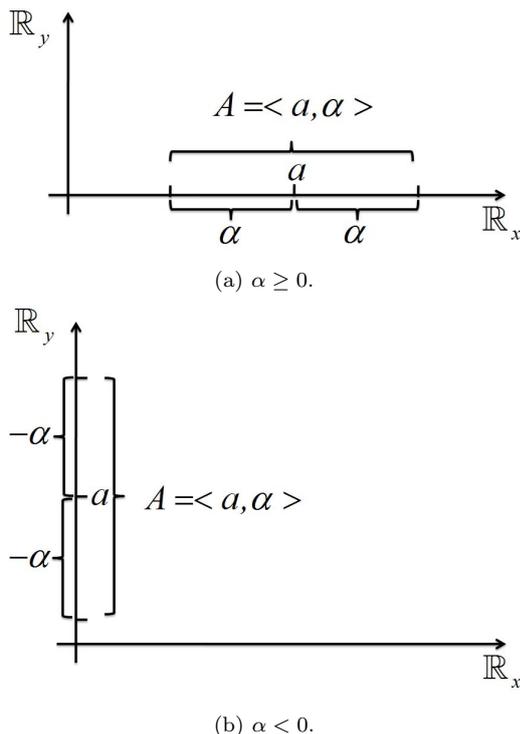
As emphasized before, the set of interval numbers under binary operations 2 and 4 as addition and multiplication operators is not a field and also it is not closed under cancellation rule. Our aim is to construct an algebraic field of intervals applying cancellation rule to define additive inverse. Here we allow interval numbers to take negative widths, and construct a new set of generalized interval as follows:

**Definition 4.** Let  $\mathbb{R}_x = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$  and  $\mathbb{R}_y = \{(0, y) \in \mathbb{R}^2 | y \in \mathbb{R}\}$ . Generalized interval number  $A = \langle a, \alpha \rangle$  is a convex closed subset of the union  $\mathbb{R}_x$  and  $\mathbb{R}_y$  such that if  $\alpha \geq 0$ ,  $\langle a, \alpha \rangle$  is an ordinary interval on  $\mathbb{R}_x$  and if  $\alpha < 0$  the  $\langle a, -\alpha \rangle$  is an ordinary interval on  $\mathbb{R}_y$ . The set of all generalized interval numbers is denoted by  $GIN(\mathbb{R})$ .

Figures 1a and 1b depict the generalized interval  $A$  when  $\alpha \geq 0$  and  $\alpha < 0$ , respectively.

Let  $A = \langle a, \alpha \rangle, B = \langle b, \beta \rangle \in GIN(\mathbb{R})$ . To define





**Figure 1.** The representation of the generalized interval  $A = \langle a, \alpha \rangle$

the addition and multiplication of  $A$  and  $B$  we have:

$$A + B = \langle a + b, \alpha + \beta \rangle. \quad (10)$$

$$A \times B = \langle a, \alpha \rangle \times \langle b, \beta \rangle = \langle ab + \alpha\beta, a\beta + b\alpha \rangle. \quad (11)$$

Then we can obtain directly:

$$\begin{aligned} -\langle a, \alpha \rangle &= \langle -1, 0 \rangle \times \langle a, \alpha \rangle = \langle -a + 0, 0 + (-1)\alpha \rangle \\ &= \langle -a, -\alpha \rangle. \end{aligned} \quad (12)$$

Thus the subtraction can be defined with the following:

$$A - B = A + (-B) = \langle a - b, \alpha - \beta \rangle. \quad (13)$$

which is the same as cancellations rule 8 for ordinary intervals.

An interval  $A = \langle a, \alpha \rangle$ , can be coupled with a traverse function  $f_A : [0, T] \mapsto \langle a, \alpha \rangle$  such that

- $T \geq 0$  is a decision period time.
- $f_A$  is one-to-one and continuous function.
- $f_A(0) = a - \alpha$  and  $f_A(T) = a + \alpha$ .

Then in addition and subtraction 10 and 13, '+' and '-' operate on locations given by traverse functions  $f_A(t)$  and  $f_B(t)$  point-by-point where  $t$  increases from 0 up to  $T$ , i.e.

$$f_{A \pm B}(t) = f_A(t) \pm f_B(t), \quad \forall t \in [0, T].$$

Therefore when  $B = A$ , since  $f_B(t) = f_A(t)$  for each

$t \in [0, T]$  we have  $A - B = 0$ , or equivalently  $A - A = 0$ .

On the other hand, if  $A, B \geq 0$  we can write:

$$\begin{aligned} f_{A \times B}(0) &= ab + \alpha\beta - (a\beta + b\alpha) = (a - \alpha)(b - \beta) \\ &= f_A(0) \times f_B(0), \end{aligned}$$

$$\begin{aligned} f_{A \times B}(T) &= ab + \alpha\beta + (a\beta + b\alpha) = (a + \alpha)(b + \beta) \\ &= f_A(T) \times f_B(T), \end{aligned}$$

thus again, we can write:

$$f_{A \times B}(t) = f_A(t) \times f_B(t), \quad \forall t \in [0, T].$$

These traverse functions are important, when uncertainty streams from pessimistically to the optimistically statuses. In these cases, the level of uncertainty in data varies monotonically and with the same trend, the corresponding traverse functions change. We name this trend as principle of uncertainty traverse.

For introducing the inverse of interval  $A = \langle a, \alpha \rangle$  where  $|a| \neq |\alpha|$ , by solving an easy mathematical exercise we obtain:

$$\langle a, \alpha \rangle \times \left\langle \frac{a}{a^2 - \alpha^2}, \frac{-\alpha}{a^2 - \alpha^2} \right\rangle = \langle 1, 0 \rangle = 1.$$

We denote the inverse of  $A$  with  $A^{-1}$ . If  $|a| = |\alpha|$ , the inverse of  $A$  is not well defined, because in these cases one of the bounds of  $A$  is zero and divide on zero is not defined.

Now for division we can define:

$$\begin{aligned} A/B &= A \times B^{-1} = \langle a, \alpha \rangle / \langle b, \beta \rangle^{-1} \\ &= \left\langle \frac{ab - \alpha\beta}{b^2 - \beta^2}, \frac{b\alpha - a\beta}{b^2 - \beta^2} \right\rangle, \end{aligned} \quad (14)$$

where  $|b| \neq |\beta|$ .

It is clear that the set of generalized interval numbers is closed under binary operations 10 and 11. The following Proposition is the main result.

**Proposition 1.** The set of generalized interval numbers under the binary operators 10 and 11 as addition and multiplication, is a field.

*Proof.* The set of generalized interval numbers is an abelian group under 10 with zero element  $(0, 0)$  and addition inverse  $\langle -a, -\alpha \rangle$  for generalized interval  $\langle a, \alpha \rangle$ . Moreover,  $GIN(\mathbb{R}) - \{\langle a, \alpha \rangle\}_{|a| \neq |\alpha|}$  is an abelian group under 11 with unit element  $\langle 1, 0 \rangle$  and multiplication inverse  $\langle a, \alpha \rangle^{-1} = \left\langle \frac{a}{a^2 - \alpha^2}, \frac{-\alpha}{a^2 - \alpha^2} \right\rangle$ .  $\square$

Therefore we obtained an interval field as a framework for solving real problems.

To define the inequality relation between two interval numbers, a lot of methods have been proposed in the literature [8]. But maybe the most convenient and directive method for comparison is ranking functions



[20] in which a ranking function  $\mathcal{R} : \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$  that maps each interval number into the real line is defined for ordering the intervals. Thus by a natural order on real numbers we can compare interval numbers easily as follows:

$$\begin{aligned} A \geq_{\mathcal{R}} B & \text{ if and only if } \mathcal{R}(A) \geq \mathcal{R}(B), \\ A >_{\mathcal{R}} B & \text{ if and only if } \mathcal{R}(A) > \mathcal{R}(B), \\ A =_{\mathcal{R}} B & \text{ if and only if } \mathcal{R}(A) = \mathcal{R}(B). \end{aligned}$$

We argue this approach in this article and use a modification of Hashemi et al.' idea [3] which can treat with intervals pessimistically as well as optimistically. For this aim we give the following definition.

**Definition 5.** Let  $A = \langle a, \alpha \rangle$  and  $B = \langle b, \beta \rangle$  be two given intervals. For each couple real positive numbers  $(k, l)$ , less than or equal relation  $\leq_{k,l}$  is defined as following:

$$\langle a, \alpha \rangle \leq_{k,l} \langle b, \beta \rangle,$$

if and only if

$$k.(a - \alpha) + l.(a + \alpha) \leq k.(b - \beta) + l.(b + \beta),$$

where  $\leq$  means the common total order on real numbers  $\mathbb{R}$ .

**Property 1.** In the ordering definition 5, two important parameters  $k$  and  $l$  are the importance of the center and the spread of interval in comparison process, respectively.

This property, easily, permit us to model risk averse, risk neutral, and risk seeking decision makers [3].

**Proposition 2.** Order relation  $\leq_{k,l}$  on interval numbers is a reflexive and transitive relation.

**Corollary 1.** The order in definition 5 can be defined applying the following linear ranking function:

$$\mathcal{R} : \text{GIN}(\mathbb{R}) \mapsto \mathbb{R}$$

$$\mathcal{R}(\langle a, \alpha \rangle) = k.(a - \alpha) + l.(a + \alpha),$$

Therefore, this order belongs to the class of ranking function orders [20].

This order is not a total order. For providing this property, Hashemi et al. [3] limited the choice of  $k$  and  $l$  using non-algebraic numbers [26]. For example the ratio of circle circumference to its diameter denoted with  $\pi \simeq 3.1415$  and the base of the natural logarithm ( $e \simeq 2.7182$ ) are non-algebraic. We have the following important result.

**Proposition 3.** Consider a non-algebraic real positive number  $\vartheta$ . Let

$$\begin{cases} k = q_1 \vartheta^{n_1}, \\ l = q_2 \vartheta^{n_2}, \end{cases}$$

where  $q_1, q_2$  are two positive rational numbers and  $n_1 \neq n_2$  are nonnegative integer numbers. Then  $\leq_{k,l}$  on  $\mathfrak{T}_{\mathbb{Q}} = \{\langle a, \alpha \rangle | a, \alpha \in \mathbb{Q}\}$  is a total order.

*Proof.* See Proposition (2.8) in Hashemi et al. [3].  $\square$

**Property 2.** Let  $A = \langle a, \alpha \rangle$  and  $B = \langle b, \beta \rangle$  be in  $\mathfrak{T}_{\mathbb{Q}}$ , then

$$A = B, \tag{15}$$

if and only if

$$k.(a - \alpha) + l.(a + \alpha) = k.(b - \beta) + l.(b + \beta), \tag{16}$$

if and only if

$$a = b, \quad \alpha = \beta, \tag{17}$$

where  $k$  and  $l$  satisfy the assumptions of Proposition 3.

## 4 Interval linear programming

In this section we consider three interval models:

- Linear programming with interval cost function and real variables.
- Linear programming with interval coefficients and real variables.
- Linear programming with interval coefficients and variables.

We show the efficiency of our scheme in solving the corresponding linear programs in comparison with the following previous works on interval numbers:

- Pessimistic order relation.

$$A \preceq_{pess} B \text{ iff } a + \alpha \leq b - \beta \tag{18}$$

- Optimistic order relation.

$$A \preceq_{opt} B \text{ iff } a - \alpha \leq b + \beta \tag{19}$$

- Adamo's order relation. [27]

$$A \preceq_{AO} B \text{ iff } a + \alpha \leq b + \beta \tag{20}$$

- Ishibuchi and Tanaka's order relation [22].

$$\begin{cases} A \preceq_{LR} B \text{ iff } a - \alpha \leq b - \beta \\ \quad \text{and } a + \alpha \leq b + \beta, & (a) \\ A \preceq_{mw} B \text{ iff } a \leq b \\ \quad \text{and } \alpha \geq \beta, & (b) \end{cases} \tag{21}$$

### 4.1 LP with interval cost function

We present the following example to reveal the efficiency of our scheme. The behind idea of this discussion is tested by Hashemi et al. [3] on a special example of network flow problems.

**Example 1.** Consider the following interval problem:



**Table 1.** The optimal solution of Example 1 applying the proposed approach for some settings of  $(k, l)$ .

$(k, l)$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	Optimal Value
$(1, 100\pi)$	0	0	166.67	0	0	200.00	$\langle 2666.67, 366.67 \rangle$
$(100\pi, 1)$	180.00	20.00	0	0	120.00	0	$\langle 920.00, 1500.00 \rangle$
$(\pi/2, 1)$	0	200.00	100.00	0	0	0	$\langle 1200.00, 700.00 \rangle$
$(\pi/7, 1)$	200.00	0	166.67	0	0	0	$\langle 1466.67, 566.67 \rangle$
$(\pi/19, 1)$	0	0	166.67	0	0	200.00	$\langle 2666.67, 366.67 \rangle$

$$\begin{aligned} \min \quad & \langle 4, 2 \rangle x_1 + \langle 4, 3 \rangle x_2 + \langle 4, 1 \rangle x_3 + \langle 6, 5 \rangle x_4 \\ & + \langle 1, 9 \rangle x_5 + \langle 10, 1 \rangle x_6, \end{aligned} \quad (22)$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_4 + x_6 & \geq 200, \\ x_1 + 3x_3 + x_5 & \geq 300, \\ x_2 + 3x_3 - x_4 + 4x_5 & \geq 500, \\ x_1, \dots, x_6 & \geq 0. \end{aligned}$$

The result for five couple of  $(k, l)$  by our approach is presented in Table 1.

This table shows that the effect of increasing in the values of  $k$  or  $l$ , is similar. Thus choosing the appropriate values for these parameters is not very hard work. Also, Table 1 illustrates that for a fixed  $l = 1$  the center of optimal values strictly increases by decreasing the value of  $k$ .

In addition, all of the feasible extreme points of this problem and their corresponding objective values are mentioned in Table 2.

For comparing the results, we sort the extreme solutions of problem based on different ordering. For this aim, for each ordering, we give a list of indices of solutions such as  $\{t_1, \dots, t_N\}$  ( $N$  is not necessarily equal to the number of solutions) such that for each  $i = 1, \dots, N - 1$ , the objective value of  $(t_i)^{th}$  solution is less than the objective value of  $(t_{i+1})^{th}$  solution based on the considered order. In Table 3, pessimistic order 18, optimistic order 19, Adamo's order 20 and finally Ishibuchi and Tanaka's orders 21.a and 21.b are compared. As you see, the possibility of inserting a solution in the list depends on the previous inserted solution. This causes to consider only solutions which can be compared together.

Table 3 reveals that our proposed scheme can find solutions similar to other ordering when the importance weights  $k$  and  $l$  are changed. Also the pessimistic order 18 and Ishibuchi and Tanakas' order 21.a and 21.b cannot rank solutions totally, which is a drawback of these methods. However the results of Adamo's order 20 is similar to the Ishibuchi and Tanakas' or-

**Table 2.** The feasible extreme solutions (points) of Example 1 and their objective values.

Index	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	Objective Value
1	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
2	180	20	0	0	120	0	$\langle 920, 1500 \rangle$
3	300	500	0	0	0	0	$\langle 3200, 2100 \rangle$
4	200	0	166.67	0	0	0	$\langle 1466.67, 566.67 \rangle$
5	166.67	0	0	33.34	133.34	0	$\langle 1000, 1700 \rangle$
6	175	0	0	0	125	25	$\langle 1075, 1500 \rangle$
7	200	0	0	0	125	0	$\langle 925, 1525 \rangle$
8	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
9	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
10	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
11	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
12	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
13	0	200	100	0	0	0	$\langle 1200, 700 \rangle$
14	0	200	0	0	300	0	$\langle 1100, 3300 \rangle$
15	0	0	55.56	200	133.34	0	$\langle 1555.56, 2255.56 \rangle$
16	0	0	233.34	200	0	0	$\langle 2133.34, 1233.34 \rangle$
17	0	0	77.78	0	66.67	200	$\langle 2377.78, 877.78 \rangle$
18	0	0	166.67	0	0	200	$\langle 2666.67, 366.67 \rangle$
19	0	0	0	700	300	0	$\langle 4500, 6200 \rangle$
20	0	0	0	200	300	0	$\langle 1500, 3700 \rangle$
21	0	0	0	0	300	200	$\langle 2300, 2900 \rangle$

der 21.a in the starting indices of solution lists (index 1 to 8) and all of them are predicted by our scheme, fortunately.

#### 4.2 LP with interval coefficients and real variables

In this subsection, we generalize the Example 1, by some new assumptions to present the new concept of the efficiency of the proposed scheme.

**Example 2.** Consider the following interval linear programming problem with real variables:



**Table 3.** The sorted list of extreme solutions applying different orders 18, 19, 20 and 21

Order relation	Sorted list	
Pessimistic order 18	1	18
Optimistic order 19	21	20 19 18 17 16 15
	14	13 12 11 10 9 8
	7	6 5 4 3 2 1
Adamo’s order 20	13	12 11 10 9 8 1
	4	2 7 6 5 18 17
	16	15 14 21 20 3 19
Ishibuchi and Tanaka’s order 21.a	13	12 11 10 9 8 1
	4	5 7 6 14 15 16
	20	21 3
Ishibuchi and Tanaka’s order 21.b	2	5 7 6 14 13 12
	11	10 9 8 1 4 20
	21	18

$$\begin{aligned} \min \quad & \langle 4, 2 \rangle x_1 + \langle 4, 3 \rangle x_2 + \langle 4, 1 \rangle x_3 + \langle 6, 5 \rangle x_4 \\ & + \langle 1, 9 \rangle x_5 + \langle 10, 1 \rangle x_6, \end{aligned} \tag{23}$$

s.t.

$$\begin{aligned} & \langle 1, 0.5 \rangle x_1 + \langle 1, 1 \rangle x_2 + x_4 + x_6 \geq \langle 200, 10 \rangle, \\ & \langle 2, 1 \rangle x_1 + \langle 3, 2 \rangle x_3 + x_5 \geq \langle 300, 50 \rangle, \\ & \langle 1, 1 \rangle x_2 + \langle 3, 2 \rangle x_3 - \langle 2, 1 \rangle x_4 + \langle 4, 3 \rangle x_5 \geq \langle 500, 40 \rangle, \\ & x_1, \dots, x_6 \geq 0. \end{aligned}$$

The constraints of this problem may be rewritten as follows:

$$\left\{ \begin{aligned} & \langle x_1 + x_2 + x_4 + x_6, 0.5x_1 + x_2 \rangle \geq \langle 200, 10 \rangle, \\ & \langle 2x_1 + 3x_3 + x_5, x_1 + 2x_3 \rangle \geq \langle 500, 40 \rangle, \\ & \langle x_2 + 3x_3 - 2x_4 + 4x_5, x_2 + 2x_3 - x_4 + 3x_5 \rangle \\ & \quad \geq \langle 500, 40 \rangle, \end{aligned} \right.$$

or by utilizing our ordering 5 with some fixed couples of  $(k, l)$  satisfying in Proposition 3, we have

$$\left\{ \begin{aligned} & (k + 0.5l)x_1 + (k + l)x_2 + kx_4 + kx_6 \\ & \quad \geq 200k + 10l, \\ & (2k + l)x_1 + (3k + 2l)x_3 + kx_5 \\ & \quad \geq 500k + 40l, \\ & (k + l)x_2 + (3k + 2l)x_3 - (2k + l)x_4 + (4k + 3l)x_5 \\ & \quad \geq 500k + 40l, \end{aligned} \right.$$

Therefore this case is transformed to the previous case which is presented in Example 1.

### 4.3 LP with interval coefficients and variables

In the bellow, a fully interval version of linear programming is introduced. This model can handle the uncertainty concepts in framework of problem and in decision maker’s solutions.

**Example 3.** Consider the following interval problem with interval variables:

$$\begin{aligned} \min \quad & \langle 4, 2 \rangle \times \langle x_1, \xi_1 \rangle + \langle 4, 3 \rangle \times \langle x_2, \xi_2 \rangle + \langle 4, 1 \rangle \times \langle x_3, \xi_3 \rangle \\ & + \langle 6, 5 \rangle \times \langle x_4, \xi_4 \rangle + \langle 1, 9 \rangle \times \langle x_5, \xi_5 \rangle + \langle 10, 1 \rangle \times \langle x_6, \xi_6 \rangle, \end{aligned} \tag{24}$$

s.t.

$$\begin{aligned} & \langle 1, 0.5 \rangle \times \langle x_1, \xi_1 \rangle + \langle 1, 1 \rangle \times \langle x_2, \xi_2 \rangle + x_4 + x_6 \geq \langle 200, 10 \rangle, \\ & \langle 2, 1 \rangle \times \langle x_1, \xi_1 \rangle + \langle 3, 2 \rangle \times \langle x_3, \xi_3 \rangle + x_5 \geq \langle 300, 50 \rangle, \\ & \langle 1, 1 \rangle \times \langle x_2, \xi_2 \rangle + \langle 3, 2 \rangle \times \langle x_3, \xi_3 \rangle - \langle 2, 1 \rangle \times \langle x_4, \xi_4 \rangle \\ & \quad + \langle 4, 3 \rangle \times \langle x_5, \xi_5 \rangle \geq \langle 500, 40 \rangle, \\ & x_1 - \xi_1, \dots, x_6 - \xi_6 \geq 0. \end{aligned}$$

By arithmetic operators 10 and 11, this problem simplifies as follows:

$$\begin{aligned} \min \quad & \langle 4x_1 + 2\xi_1 + 4x_2 + 3\xi_2 + 4x_3 + \xi_3 + 6x_4 + 5\xi_4 + x_5 + 9\xi_5 \\ & + 10x_6 + \xi_6, 2x_1 + 4\xi_1 + 3x_2 + 4\xi_2 + x_3 + 4\xi_3 + 5x_4 \end{aligned}$$



$$+6\xi_4 + 9x_5 + \xi_5 + x_6 + 10\xi_6), \quad (25)$$

s.t.

$$\left\{ \begin{array}{l} \langle x_1 + 0.5\xi_1 + x_2 + \xi_2 + x_4 + x_6, 0.5x_1 + \xi_1 \\ \quad + x_2 + \xi_2 \rangle \geq \langle 200, 10 \rangle \\ \langle 2x_1 + \xi_1 + 3x_3 + 2\xi_3 + x_5, x_1 + 2\xi_1 + 2x_3 \\ \quad + 3\xi_3 \rangle \geq \langle 300, 50 \rangle, \\ \langle x_2 + \xi_2 + 3x_3 + 2\xi_3 - 2x_4 - \xi_4 + 4x_5 + 3\xi_5, x_2 + \\ \quad \xi_2 + 2x_3 + 3\xi_3 - x_4 - 2\xi_4 + 3x_5 + 4\xi_5 \rangle \geq \langle 500, 40 \rangle, \end{array} \right.$$

The objective function can be rewritten as bellow:

$$\begin{aligned} \min \quad & \langle 4, 2 \rangle x_1 + \langle 4, 3 \rangle x_2 + \langle 4, 1 \rangle x_3 + \langle 6, 5 \rangle x_4 + \langle 1, 9 \rangle x_5 \\ & + \langle 10, 1 \rangle x_6 + \langle 2, 4 \rangle \xi_1 + \langle 3, 4 \rangle \xi_2 + \langle 1, 4 \rangle \xi_3 + \langle 5, 6 \rangle \xi_4 \\ & + \langle 9, 1 \rangle \xi_5 + \langle 1, 10 \rangle \xi_6. \end{aligned}$$

Also applying the ordering 5 with some fixed couples of  $(k, l)$  fulfilling in Proposition 3, we can transform the constraint as follows:

$$\left\{ \begin{array}{l} (k + 0.5l)x_1 + (0.5k + l)\xi_1 + (k + l)x_2 + (k + l)\xi_2 \\ \quad + kx_4 + kx_6 \geq 200k + 10l \\ (2k + l)x_1 + (k + 2l)\xi_1 + (3k + 2l)x_3 + (2k + 3l)\xi_3 \\ \quad + kx_5 \geq 300k + 50l, \\ (k + l)x_2 + (k + l)\xi_2 + (3k + 2l)x_3 + (2k + 3l)\xi_3 \\ \quad - (2k + l)x_4 - (k + 2l)\xi_4 + (4k + 3l)x_5 \\ \quad + (3k + 4l)\xi_5 \geq 500k + 40l. \end{array} \right.$$

Thus a problem with interval objective function and real restriction is obtained and the result of Example 1 is applicable for this state, too. Only note that this problem includes two series of variables.

#### 4.4 LP with fuzzy numbers

A linear programming with fuzzy parameters can be used to model a real problem under uncertainty. Such models are very applicable for long-term decision making. Fuzzy optimization models can be solved differently. In [20], three different approaches for fuzzy ordering have been discussed which can be considered to produce algorithms for fuzzy optimization. Also each fuzzy number  $\tilde{A}$  can be represented with its  $\alpha$ -cuts directly, see e.g., [7]. Then for a fixed certainty degree, it is possible to consider one of the  $\alpha$ -cuts in which all of the members in  $\alpha$ -cut corresponds a possibility degree greater than  $\alpha$ . Since the  $\alpha$ -cut for a fuzzy number is an interval, all of the concepts given in the previous subsections are valid.

## 5 Application

Assume there are  $n$  projects for investment. Let  $\langle c_i, \gamma_i \rangle$  is the profit of  $i^{th}$  project considering all of the information and experiments. We need to maximize the following interval value objective function:

$$\max z = \sum_{i=1}^n \langle c_i, \gamma_i \rangle x_i, \quad (26)$$

where non-negative variable  $x_i$  is the amount of capital which is invested in project  $i$ .

Let risk of investment in  $i^{th}$  project is  $\langle r_i, \alpha_i \rangle$ . If the bound for risk is given by  $\langle t_i, \beta_i \rangle$ . This dictate to consider the following constraint:

$$\sum_{i=1}^n \langle r_i, \alpha_i \rangle x_i \leq \langle t_i, \beta_i \rangle, \quad (27)$$

This problem needs to solve under the following budget restriction:

$$\sum_{i=1}^n x_i \leq B, \quad (28)$$

The approach of this paper can be followed to solve this uncertain problem.

## 6 Conclusion and future direction

It is reasonable in each area of mathematics that mathematician want to work on an appropriate algebraic structure such as group or field. We showed our known intervals cannot create a field by classic definition of operators. Therefore we define addition and multiplication in direct form and obtain two formulas for subtraction and division. In addition, we interpret the operators with level of uncertainty. These new operators can be used to produce some interval with negative width which can be appear in solution of interval systems. We inserted these new intervals to the set of classic intervals and showed that this new set is a field. To define operators on this interval field, we propose principle of uncertainty traverse which provides an opportunity to extend interval operators as the same as operators on real numbers. Then by employing non-algebraic numbers, we extend a total order on a sufficiently large subset of interval field. Then we solve three versions of interval linear programming and compare our results with other orderings. These examples demonstrate the efficiency of our schemes. Introducing on the duality concepts of interval linear programming based on the mentioned preliminaries is left to the next works.



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