Solution of the interval linear programming problems with respect to the pessimistic and optimistic perspectives

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Abstract

This paper deals with linear programming problem with interval numbers as coefficients to exhibit with uncertainty concepts. Since, the set of known intervals is not a field, we define a new interval type numbers and prove that the set including such interval numbers is an algebraic field. Applying a total order on this field, we propose an implementable scheme for interval linear programming problem and find solution for this problem with respect to the pessimistic and optimistic attitudes. The numerical experiments are given to demonstrate the efficiency of proposed scheme in comparison with the previous established works. The approach in this paper can be generalized for fuzzy linear programming taking the fuzzy cuts into account.

Keywords: Interval Field, Interval Linear Programming, Total Ordering.

1 Introduction

For some considerable time, linear programming (LP) has been one of the operational research techniques, which has been widely used and got many achievements in both applications and theories [15, 28]. However the strict requirement of LP is that the data must be well defined and precise, which is often impossible in real decision problems. The traditional way to evaluate any imprecision in the parameters of an LP model is through a post-optimization analysis, with the help of sensitivity analysis and parametric programmin. However, none of these methods is suitable for an overall analysis of the effects of imprecision in parameters. Another way to handle imprecision is to model it in stochastic programming problems according to the probability theory. A third way to cope with imprecision is to resort to the theory of interval mathematics or fuzzy sets, which give the conceptual and theoretical framework for dealing with complexity and uncertainty [27, 12, 31]. Interval mathematics started in 1950s and came more applicable in programming soon. They can handel uncertainty aspects according to the statistical analysis which can estimate the quantities bounds. Nowadays very researchers focus on this type of programming in place of exact versions. For some information and application about interval programming interested reader may address

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to [10, 16, 8]. Also, in fuzzy mathematics, intervals are revealed as fuzzy cuts [9], so interval results are the base for these cases, too. Besides, since the membership functions of fuzzy quantities can not find easily, interval programming is a powerful alternative instead of fuzzy versions, certainly [26, 12]. Interval numbers is important in numerical analysis too, which is studying on the methods for solving interval linear systems, interval interpolations, and other scope of uncertain computations [22, 10, 11, 19, 23]. Although the most basic results on interval analysis published by Moore [22], however with definition of complex intervals by Boche [3], this set was very extended. A complex interval can represent by a rectangle or a circle in the complex plane, i.e. such number is shown by an interval real part and an interval imaginary part, or by a triple consist of the center, low radius and up radius respectively, as similar as a disk [24]. Then Petkovic et al. [24] collected arithmetic on complex intervals. In similar scope, Ramot et al. [25] introduced complex fuzzy numbers too. In their work membership functions were as complex valued functions.

In parallel to these extensions, Kandel et al. [17] and Friedman et al. [7] in solving fuzzy linear systems found some contradiction solutions, which their right limits are less than left limits. This contradiction is appeared from the gap in fuzzy numbers. Respect to the practitioners' attempt for finding appropriate solutions to each system, Friedman et al. [7] accepted those unrealistic solutions and named them as weak fuzzy solutions. Allahviranloo [2] used this terminology, too. Chang and Lee [4] faced to the same problem in fuzzy regression and proposed fuzzy numbers with spreads unrestricted in sign.

We believe, this gap in fuzzy numbers is a result of similar gap in intervals. So we must fill the set of intervals with new numbers at first, which simplify the calculations. Our approach is to define arithmetic operators on interval numbers as well as has published by Ganesan and Veeramani [8]. Also, in this paper we follow the appearance process of negative integers, i.e. we define the addition and multiplication on interval numbers and obtain , subtraction and division for them and show that some interval with negative spreads can be appeared. Immediately, we define the set of extended real intervals and prove that this new set is a field respect to its addition and multiplication. For demonstrating the efficiency of this scheme, we argue linear programming on this interval field.

The proposed methods for solving the same programs, usually, are constructed on a rank for comparing interval or fuzzy data, which are varied in essays [26, 29]. In ordering comments, Moore [21] is the first author which extended $'<'$ on real line and $'\subseteq'$ on the sets to create two transitive order relations over intervals. Ishibuchi and Tanaka [14] as an advance over Moore, suggested two order relations $\leq_{LR}$ and $\leq_{MW}$ on them. However these ordering were not total order. Kundu [20] defined a fuzzy preference relationship between two interval and used it to find optimal decision. More differently Sengupta and Pal [26] suggested two index for comparing two interval numerically. But almost each approach pay attention to the first or the end points of intervals. In this paper we follow Hashemi et al. idea [12] for minimum interval cost flow problem. The high advantage of this scheme is its capability for taking risk personality in decision making processes. We show the efficiency of this approach for interval linear programming in this paper. The corresponding idea may be extended for multiobjective variant.

The rest of paper is organized as follows:

In the next section, we review on the classic theory of arithmetic operators on intervals with representation of some drawbacks in that scope. In section 3 we introduce a generalization of intervals which produces a field of intervals. Then a total order by pessimistically up to optimistically behavior is introduced. An interval linear programming with three groups of assumption are solved in section 4. Section 5 ends the paper with conclusion and future directions.

2 Interval Numbers

An interval number A is the set of all real numbers x, such that $a_L \leq x \leq a_R$, where a_L and a_R are left and right limits of the interval A. We denote the set of interval numbers by IN. The interval number A is usually denoted by $A = [a_L, a_R]$. If $a_L = a_R$, then A is a real number. Such an interval is said to be degenerate. Interval A is alternatively represented as $A = \langle a, \alpha \rangle$, where $a = \frac{a_L + a_R}{2}$ and $\alpha = \frac{a_R - a_L}{2}$ are center and width of interval number A, respectively. In this paper, we use the latter notation for representation of interval numbers. An interval $A = \langle a, \alpha \rangle$ is said to be nonnegative if $a - \alpha \geq 0$ and nonpositive if $a + \alpha \leq 0$. There are two important topics in the real world applications of interval numbers, namely arithmetic operations and ordering of interval numbers. Traditionally, arithmetic operations on interval numbers are defined by the extension principle [22, 10]. Let $f : \Re \times \Re :\rightarrow \Re$ be a binary operation over real numbers. Then it can be extended to the operation over interval numbers. If A and B be two interval numbers and $* \in \{+, -, ., /\}$ be a binary operation on the set of real numbers, then using the extension principle, the binary operation $*$ over interval numbers A and B are defined as follows [22]:

$$A * B = \{a * b | a \in A, b \in B\}. \tag{1}$$

In the case of division it is assumed that 0 not exists in B.

If $A = \langle a, \alpha \rangle$ and $B = \langle b, \beta \rangle$, then the extended addition, substraction, multiplication, and division are derived as

$$\langle a, \alpha \rangle + \langle b, \beta \rangle = \langle a + b, \alpha + \beta \rangle, \tag{2}$$

$$\langle a, \alpha \rangle - \langle b, \beta \rangle = \langle a - b, \alpha + \beta \rangle, \tag{3}$$

$$\langle a, \alpha \rangle \times \langle b, \beta \rangle = \langle (d + c)/2, (d - c)/2 \rangle, \tag{4}$$

$$\langle a, \alpha \rangle / \langle b, \beta \rangle = \langle (f + e)/2, \langle f - e)/2 \rangle, \tag{5}$$

where

$$c = \min\{(a - \alpha)(b - \beta), (a - \alpha)(b + \beta), (a + \alpha)(b - \beta), (a + \alpha)(b + \beta)\},$$

$$d = \max\{(a - \alpha)(b - \beta), (a - \alpha)(b + \beta), (a + \alpha)(b - \beta), (a + \alpha)(b + \beta)\},$$

$$e = \min\{\frac{a - \alpha}{b - \beta}, \frac{a - \alpha}{b + \beta}, \frac{a + \alpha}{b - \beta}, \frac{a + \alpha}{b + \beta}\},$$

$$f = \max\{\frac{a - \alpha}{b - \beta}, \frac{a - \alpha}{b + \beta}, \frac{a + \alpha}{b - \beta}, \frac{a + \alpha}{b + \beta}\}.$$

If both of A and B are nonnegative or nonpositive intervals, then extended multiplication and division are simplified as

$$\langle a, \alpha \rangle \times \langle b, \beta \rangle = \langle ab + \alpha\beta, a\beta + b\alpha \rangle, \tag{6}$$

(7) $$\langle a, \alpha\rangle / \langle b, \beta\rangle = \langle \frac{ab+\alpha\beta}{b^2-\beta^2}, \frac{a\beta+b\alpha}{b^2-\beta^2}\rangle.$$

The extended interval operations (2)-(5) have been used in solving fuzzy linear system of equations e.g. in Gaussian Elimination [11]. In [23, 10] interval interpolation and other numerical analysis topics with extended interval operations were developed.

Groups and fields are familiar objects to us. These are all sets of elements with additional structure (that is, various ways of combining elements to produce an element of the set). Studying this finer structure is the key to many deep facts in number theory.

Definition 2.1 A group is a set G which is closed under a binary operation $*$ (that is, for any $x, y \in G, x * y \in G$) and satisfies the following properties:

1. Identity There is an element $e \in G$, such that for every $x \in G, e * x = x * e = x$.
2. Inverse For every $x \in G$ there is an element $y \in G$ such that $x * y = y * x = e$, where again e is the identity.
3. Associativity The following identity holds for every $x, y, z \in G$:

$$x * (y * z) = (x * y) * z.$$

Definition 2.2 A group is said to be abelian if $x * y = y * x$ for every $x, y \in G$.

Definition 2.3 A field is a set F which is closed under two binary operations $+$ and $*$ (called addition and multiplication) such that

1. F is an abelian group under $+$ and
2. $F - \{0\}$ (the set F without the additive identity 0) is an abelian group under $*$.
3. For each $x, y, z \in F$ we have:

$$x * (y + z) = x * y + x * z.$$

The identity elements under addition and multiplication operations are called zero and unit elements, respectively.

It is clear that the interval $\langle 0, 0\rangle$ is the zero element for IN, but there is no addition inverse for interval numbers with positive width. Therefore, the set of interval numbers under binary operation (2) is not a group. Therefore, the set IN under binary operations (2) and (4) as addition and multiplication operators is not a field. So we cannot solve system of interval equations efficiently. For example, we consider the simple example presented by Hansen [10]. Assume that n intervals $A_i, i = 1, \ldots, n$ are given and for each $i = 1, \ldots, n$ we want the sum of all except the i^{th} interval. Suppose that we first compute the sum

$$S_1 = A_2 + \ldots + A_n.$$

Afterwards, we want the sum

$$S_2 = A_1 + A_3 + \ldots + A_n.$$

Instead of calculation the sum S_2 directly, we are going to use the previous result. Note that $S_2 = S_1 + A_1 - A_2$. So we can compute S_2 by adding A_1 to S_1 and then cancelling A_2 from the result by subtraction. But $A_2 - A_2 = \langle a_2 - a_2, \alpha_2 + \alpha_2 \rangle = \langle 0, 2\alpha_2 \rangle$ is not the (degenerate) zero interval. Therefore, we cannot cancel A_2 from S_1 simply by subtracting unless A_2 is real numbers.

Instead of subtracting using extended interval substraction, Hansen [10] used the special cancellation rule as follows:

$$\langle a, \alpha \rangle \backslash \langle b, \beta \rangle = \langle a - b, \alpha - \beta \rangle. \tag{8}$$

This operator is similar as Hukuhara's difference for fuzzy numbers [13, 5].
Now let us consider the following simple interval linear equation.

$$\langle 35, 2.5 \rangle + \langle x, y \rangle = \langle 50, 3.5 \rangle. \tag{9}$$

In order to solve equation (9), we may write the following relation

$$\langle x, y \rangle = \langle 50, 3.5 \rangle - \langle 35, 2.5 \rangle = \langle 15, 6 \rangle,$$

while this solution don't satisfy in the equation (9). But if instead of extended substraction we use the cancellation rule (8), then we obtain the desired solution $\langle 15, 1 \rangle$. Therefore, cancellation is more applicable in these cases in comparison with extended interval subtraction. On the other hand, let us given the interval numbers $A = \langle 150, 10 \rangle$ and $B = \langle 125, 5 \rangle$. Then using cancellation rule (8) to compute $A \backslash B$, we obtain $\langle 25, -5 \rangle$. Note that $\langle 25, -5 \rangle$ is not an interval number since the width of interval numbers must be nonnegative. So cancellation rule is not closed on interval numbers. In the next section we fill this blank.

3 Generalized Interval Numbers

As emphasized before, the set of interval numbers under binary operations (2) and (4) as addition and multiplication operators is not a field and is not closed under cancellation rule. Our aim is to construct an algebraic structure of interval numbers in which it will be a field and cancellation rule is closed. In order to do that, we generalize interval numbers into a larger set.

From the definition of interval numbers, the width of interval numbers should be nonnegative. Here we allow interval numbers to take negative widths, and construct a new set as follows:

Definition 3.1 *Let $\mathbb{R}_x = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ and $\mathbb{R}_y = \{(0, y) \in \mathbb{R}^2 | y \in \mathbb{R}\}$. Generalized interval number $A = \langle a, \alpha \rangle$ is a convex closed subset of the union $\mathbb{R}_x$ and $\mathbb{R}_y$ such that if $\alpha \geq 0$, $\langle a, \alpha \rangle$ is an ordinary interval on $\mathbb{R}_x$ and if $\alpha < 0$ the $\langle a, -\alpha \rangle$ is an ordinary interval on $\mathbb{R}_y$. The set of all generalized interval numbers is denoted by $GIN(\mathbb{R})$.*

Figure 1.(a) and 1.(b) depict the generalized interval A respect to $\alpha \geq 0$ and $\alpha < 0$, respectively. Let $A = \langle a, \alpha \rangle, B = \langle b, \beta \rangle \in GIN(\mathbb{R})$. We now define the addition and multiplication of A and B as follows, respectively:

$$A + B = \langle a + b, \alpha + \beta \rangle. \tag{10}$$

$$A \times B = \langle a, \alpha \rangle \times \langle b, \beta \rangle = \langle ab + \alpha\beta, a\beta + b\alpha \rangle. \tag{11}$$

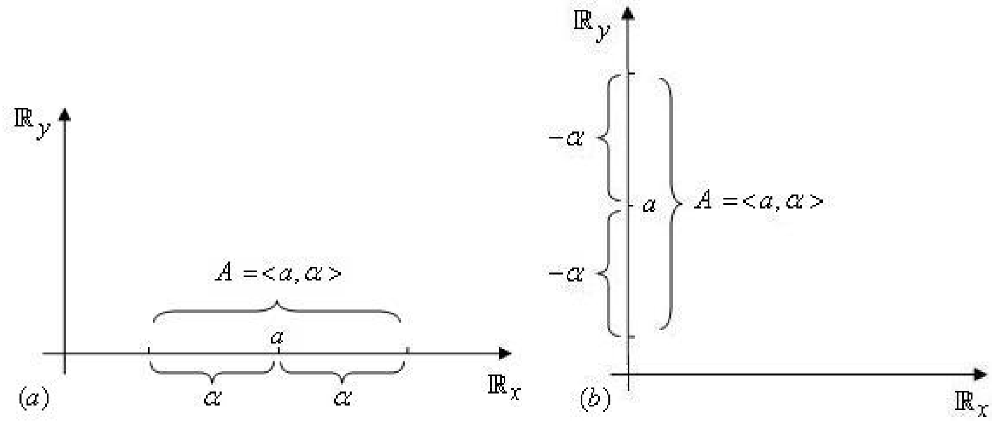

Figure 1: (a) and (b) are shown the interval $A = \langle a, \alpha \rangle$ where $\alpha \geq 0$ and $\alpha < 0$, respectively.

Then we can obtain directly:

(12) $$-\langle a, \alpha \rangle = \langle -1, 0 \rangle \times \langle a, \alpha \rangle = \langle -a + 0, 0 + (-1)\alpha \rangle = \langle -a, -\alpha \rangle.$$

Thus the subtraction can be defined with the following:

(13) $$A - B = A + (-B) = \langle a - b, \alpha - \beta \rangle.$$

which is the same as cancellations rule (8) for ordinary intervals.
Please note that the corresponding addition and subtraction in (10 and 13) have nice interpretations. To explain this aspect in familiar way, for a fix time parameter $T \geq 0$ and each interval $A = \langle a, \alpha \rangle$, consider the location function $f_A : [0, T] \mapsto \langle a.\alpha \rangle$ such that

- f_A be one-to-one and continuous function.
- $f_A(0) = a - \alpha$ and $f_A(T) = a + \alpha$.

Then in addition and subtraction (10 and 13), $'+'$ and $'-'$ operate on location functions $f_A(t)$ and $f_B(t)$ point-by-point where t increases from 0 up to T, i.e.

$$f_{A \pm B}(t) = f_A(t) \pm f_B(t), \qquad \forall t \in [0, T].$$

Therefore when $B = A$, sine $f_B(t) = f_A(t)$ for each $t \in [0, T]$ we have $A - B = 0$, or equivalently $A - A = 0$.
On the other hand, if $A, B \geq 0$ we can write:

$$f_{A \times B}(0) = ab + \alpha\beta - (a\beta + b\alpha) = (a - \alpha)(b - \beta) = f_A(0) \times f_B(0),$$

$$f_{A \times B}(T) = ab + \alpha\beta + (a\beta + b\alpha) = (a + \alpha)(b + \beta) = f_A(T) \times f_B(T),$$

thus again, we can write:

$$f_{A \times B}(t) = f_A(t) \times f_B(t), \qquad \forall t \in [0, T].$$

These location functions are important, when decision maker's perspective streams from pessimistically to the optimistically status. In these cases, the level of uncertainty in data

vary monotonically and with the same trend. The corresponding location functions appear this trend.
Moreover, for introducing the inverse of interval $A = \langle a, \alpha \rangle$ where $|a| \neq |\alpha|$, by solving an easy mathematical exercise we obtain:

$$\langle a, \alpha \rangle \times \langle \frac{a}{a^2 - \alpha^2}, \frac{-\alpha}{a^2 - \alpha^2} \rangle = \langle 1, 0 \rangle = 1.$$

We denote the inverse of A with A^{-1}. Please note that if $|a| = |\alpha|$, the inverse of A is not well defined, because in these cases one of the bounds of A is zero and divide on zero is not defined.
Now for division we can define:

$$A/B = A \times B^{-1} = \langle a, \alpha \rangle / \langle b, \beta \rangle^{-1} = \langle \frac{ab - \alpha\beta}{b^2 - \beta^2}, \frac{b\alpha - a\beta}{b^2 - \beta^2} \rangle, \tag{14}$$

where $|b| \neq |\beta|$.
It is clear that the set of generalized interval numbers is closed under binary operations (10) and (11). The following Proposition is the main result.

Proposition 3.2 The set of generalized interval numbers under the binary operators (10) and (11) as addition and multiplication, is a field.

Proof. The set of generalized interval numbers is an abelian group under (10) with zero element $\langle 0, 0 \rangle$ and addition inverse $\langle -a, -\alpha \rangle$ for generalized interval $\langle a, \alpha \rangle$. Moreover, $GIN(\mathbb{R}) - \{\langle a, \alpha \rangle\}_{|a| \neq |\alpha|}$ is an abelian group under (11) with unit element $\langle 1, 0 \rangle$ and multiplication inverse $\langle a, \alpha \rangle^{-1} = \langle \frac{a}{a^2 - \alpha^2}, \frac{-\alpha}{a^2 - \alpha^2} \rangle$.
Therefore we obtain an interval field that is a good area for solving linear systems and other mathematical computations. It is clear that by using generalized interval numbers and generalized interval operations, we don't loss any property of real and ordinary positive interval numbers. So we are not very far from real and ordinary interval mathematics.
To define the inequality relation between two interval numbers, a lot of methods was proposed in the literature [26]. But maybe the most convenient and directive method in this area is based on the concept of comparison of interval numbers by use of ranking functions [29] in witch a ranking function $\mathscr{R} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ that maps each interval number into the real line is defined for ordering the intervals. Thus by a natural order on real numbers we can compare interval numbers easily as follows:

$A \geq_{\mathscr{R}} B$ if and only if $\mathscr{R}(A) \geq \mathscr{R}(B)$,
$A >_{\mathscr{R}} B$ if and only if $\mathscr{R}(A) > \mathscr{R}(B)$,
$A =_{\mathscr{R}} B$ if and only if $\mathscr{R}(A) = \mathscr{R}(B)$.

We argue this approach in this article and use a modification of Hashemi et al.' idea [12] which can transmit from pessimistic up to optimistic case as follows:

Definition 3.3 *Let $A = \langle a, \alpha \rangle$ and $B = \langle b, \beta \rangle$ be two given intervals. For each couple real positive numbers (k, l), less than or equal relation $\leq_{k,l}$ is defined as following:*

$$\langle a, \alpha \rangle \leq_{k,l} \langle b, \beta \rangle,$$

if and only if

$$k.(a - \alpha) + l.(a + \alpha) \leq k.(b - \beta) + l.(b + \beta),$$

where $\leq$ means the common total relation on real numbers $\mathbb{R}$.

Property 3.4 *In the ordering definition (3.3), two important parameters k and l are the importance of lower and upper bounds of intervals in decision making process, respectively.*

This property, easily, permit us to model risk averse, risk neutral, and risk seeking decision maker which can be interpreted as pessimistic and optimistic status in many real problems that can be followed in many optimization techniques [12].

Proposition 3.5 *Order relation $\leq_{k,l}$ on interval numbers is a reflexive and transitive relation.*

Corollary 3.6 *The order in definition (3.3) can be defined respect to the following linear ranking function:*

$$\mathscr{R} : GIN(\mathbb{R}) \mapsto \mathbb{R}$$

$$\mathscr{R}(\langle a, \alpha \rangle) = k.(a - \alpha) + l.(a + \alpha),$$

Therefore, this ranking is belong to the class of ranking function ordering [29].

This assertion can not be proved in general case. Because we do not have the third property that mentioned for ranking functions. For getting this property we have a proposal for making a total order. Let's limit the choice of k and l. Employing non-algebraic numbers is a suitable choice [12]. Non-algebraic numbers is very important in number theory and polynomial rings. Among the hundreds of references, the reader may address Filaseta [6] for details.

Definition 3.7 *A complex number $\mathscr{X}$ is named algebraic if and only if, it is a root of a non-zero polynomial equation by integer coefficients, else named non-algebraic or transcendental.*

Example. The circumference of the circle to its diameter denoted with $\pi \simeq 3.1415$ and the base of the natural logarithm ($e \simeq 2.7182$) are non-algebraic.
Denote the set of rational numbers and positive rational numbers with $\mathbb{Q}$ and $\mathbb{Q}^+$, respectively. We have the following important result.

Proposition 3.8 *Consider non-algebraic real positive number ϑ. Let*

$$\begin{cases} k = q_1 \vartheta^{n_1}, \\ l = q_2 \vartheta^{n_2}, \end{cases}$$

where $q_1, q_2 \in \mathbb{Q}^+$ and $n_1 \neq n_2$ are nonnegative integer numbers. Then $\leq_{k,l}$ on $\mathfrak{T}_{\mathbb{Q}} = \{\langle a, \alpha \rangle | a, \alpha \in \mathbb{Q}\}$ is a total order.

Proof. The proof of this proposition is likewise to that of Proposition (2.8) in Hashemi et al. [12]. Only note that if $A = \langle a, \alpha \rangle \leq_{k,l} \langle b, \beta \rangle = B$ and $B \leq_{k,l} A$, then

$$k.(a - \alpha) + l.(a + \alpha) = k.(b - \beta) + l.(b + \beta),$$

or,

$$q_1 \pi^{n_1}.(a - b + (\beta - \alpha)) + q_2 \pi^{n_2}.(a - b + (\alpha - \beta)) = 0,$$

but $(a - b + (\beta - \alpha))$ and $(a - b + (\alpha - \beta))$ are rational; So by producing two side of equation in sufficiently large number, an equation with integer coefficients can be obtained. So

$$a - b + (\beta - \alpha) = 0 \;\; and \;\; a - b + (\alpha - \beta) = 0,$$

by summing we have $a = b$ and by substituting $a = b$ we obtain $\alpha = \beta$. $\square$

Property 3.9 *Let* $A = \langle a, \alpha \rangle$ *and* $B = \langle b, \beta \rangle$ *be in* $\mathfrak{T}_{\mathbb{Q}}$*, then*

$$(15) \qquad A = B,$$

if and only if

$$(16) \qquad k.(a-\alpha) + l.(a+\alpha) = k.(b-\beta) + l.(b+\beta),$$

if and only if

$$(17) \qquad a = b, \qquad \alpha = \beta,$$

where k *and* l *satisfy the assumptions of Proposition (3.8).*

4 Interval linear programming

In this section we consider three interval models:

- Linear programming with interval cost function and real variables.
- Linear programming with interval coefficients and real variables.
- Linear programming with interval coefficients and variables.

We show the efficiency of our scheme in solving the corresponding linear programs in comparison with nice previous works respect to the following order relations on interval numbers:

- Pessimistic order relation.

$$(18) \qquad A \preceq_{pess} B \quad \text{iff} \quad a + \alpha \leq b - \beta$$

- Optimistic order relation.

$$(19) \qquad A \preceq_{opt} B \quad \text{iff} \quad a - \alpha \leq b + \beta$$

- Adamo's order relation. [1]

$$(20) \qquad A \preceq_{AO} B \quad \text{iff} \quad a + \alpha \leq b + \beta$$

- Ishibuchi and Tanaka's order relation [14].

$$(21) \qquad \begin{cases} A \preceq_{LR} B \quad \text{iff} \quad a - \alpha \leq b - \beta \quad \text{and} \quad a + \alpha \leq b + \beta, & (a) \\ A \preceq_{mw} B \quad \text{iff} \quad a \leq b \quad \text{and} \quad \alpha \geq \beta, & (b) \end{cases}$$

4.1 LP with interval cost function

We present the following example to reveal the efficiency of our scheme. The behind idea of this discussion is tested by Hashemi et al. [12] on a special example of network flow problems.

Example 4.1 *Consider the following interval problem:*

(22) $$\min \ \langle 4,2\rangle x_1 + \langle 4,3\rangle x_2 + \langle 4,1\rangle x_3 + \langle 6,5\rangle x_4 + \langle 1,9\rangle x_5 + \langle 10,1\rangle x_6,$$

s.t.

$$x_1 + x_2 + x_4 + x_6 \geq 200,$$

$$x_1 + 3x_3 + x_5 \geq 300,$$

$$x_2 + 3x_3 - x_4 + 4x_5 \geq 500,$$

$$x_1, ..., x_6 \geq 0.$$

The result for five couple of (k, l) by our approach is presented in Table (1).

Table (1): The optimal solution of Example (4.1) respect to our approach for some setting of (k, l).

Test Index	(k,l)	x_1	x_2	x_3	x_4	x_5	x_6	Optimal Value
1	$(1, 100\pi)$	0	0	166.67	0	0	200.00	$\langle 2666.67, 366.67\rangle$
2	$(100\pi, 1)$	180.00	20.00	0	0	120.00	0	$\langle 920.00, 1500.00\rangle$
3	$(\pi/2, 1)$	0	200.00	100.00	0	0	0	$\langle 1200.00, 700.00\rangle$
4	$(\pi/7, 1)$	200.00	0	166.67	0	0	0	$\langle 1466.67, 566.67\rangle$
5	$(\pi/19, 1)$	0	0	166.67	0	0	200.00	$\langle 2666.67, 366.67\rangle$

This table is shown where we decrease one of the parameters k or l, the result is similar to the increasing the other. Thus choosing the appropriate values for these parameters is not very hard work. Also, Table (1) illustrates for a fixed $l = 1$ the center of optimal values strictly increases respect to the decreasing in k.
Now, please notice to the all of the feasible extreme points of this problem associate with their objective values which mentioned in Table (2).

Table (2): The feasible extreme points of Example (4.1) and their objective values.

Index	x_1	x_2	x_3	x_4	x_5	x_6	Objective Value
1	0	200	100	0	0	0	$\langle 1200, 700\rangle$
2	180	20	0	0	120	0	$\langle 920, 1500\rangle$
3	300	500	0	0	0	0	$\langle 3200, 2100\rangle$
4	200	0	166.67	0	0	0	$\langle 1466.67, 566.67\rangle$
5	166.67	0	0	33.34	133.34	0	$\langle 1000, 1700\rangle$
6	175	0	0	0	125	25	$\langle 1075, 1500\rangle$
7	200	0	0	0	125	0	$\langle 925, 1525\rangle$
8	0	200	100	0	0	0	$\langle 1200, 700\rangle$
9	0	200	100	0	0	0	$\langle 1200, 700\rangle$
10	0	200	100	0	0	0	$\langle 1200, 700\rangle$
11	0	200	100	0	0	0	$\langle 1200, 700\rangle$
12	0	200	100	0	0	0	$\langle 1200, 700\rangle$
13	0	200	100	0	0	0	$\langle 1200, 700\rangle$
14	0	200	0	0	300	0	$\langle 1100, 3300\rangle$
15	0	0	55.56	200	133.34	0	$\langle 1555.56, 2255.56\rangle$
16	0	0	233.34	200	0	0	$\langle 2133.34, 1233.34\rangle$
17	0	0	77.78	0	66.67	200	$\langle 2377.78, 877.78\rangle$
18	0	0	166.67	0	0	200	$\langle 2666.67, 366.67\rangle$
19	0	0	0	700	300	0	$\langle 4500, 6200\rangle$
20	0	0	0	200	300	0	$\langle 1500, 3700\rangle$
21	0	0	0	0	300	200	$\langle 2300, 2900\rangle$

For comparing the results, we construct a list of indices of solutions as $\{t_1, ..., t_N\}$ (N is not necessarily equal to the number of solutions) such that for each $i = 1, ..., N-1$, the objective value respect to $(t_i)^{th}$ index is less than the objective value respect to $(t_{i+1})^{th}$ index by employing pessimistic order (18), optimistic order (19), Adamo's order (20) and finally Ishibuchi and Tanaka's order (21.a, 21.b). Unfortunately, the mentioned orders are depend on the inserted solution, i.e. after the first inserted solution, the another solutions should be bigger than it for permitting to insert in the list. In Table (3) the sorted indices of feasible extreme solutions of Table (2) with the corresponding orders are represented.

Table (3): The sorted sequences by ordering (18), (19), (20) and [14]).

Order relation							
Pessimistic order (18)	1	18					
Optimistic order (19)	21	20	19	18	17	16	15
	14	13	12	11	10	9	8
	7	6	5	4	3	2	1
Adamo's order (20)	13	12	11	10	9	8	1
	4	2	7	6	5	18	17
	16	15	14	21	20	3	19
Ishibuchi and Tanaka's order (21.a)	13	12	11	10	9	8	1
	4	5	7	6	14	15	16
	20	21	3				
Ishibuchi and Tanaka's order (21.b)	2	5	7	6	14	13	12
	11	10	9	8	1	4	20
	21	18					

Table (3) reveal that our proposed scheme can find solutions respect to each of the ordering except optimistic order which choose 21^{st} solution with objective value $\langle 2300, 2900\rangle$. But please note that the optimistic bound of this solution is $2900 - 2300 = 600$ and our scheme find $\langle 920, 1500\rangle$ with the optimistic bound $1500 - 920 = 580$ which illustrate the closeness of theses optimistic bounds. Also the pessimistic order (18) and Ishibuchi and Tanakas' order (21.a) and (21.b) can not rank solutions totally, which is a drawback of that methods. However the results of Adamo's order (20) is similar to the Ishibuchi and Tanakas' order (21.a) in the start of comparisons (index 1 to 8) and all of them are predicted by our scheme, fortunately.

4.2 LP with interval coefficients and real variables

In this subsection, we generalize the Example (4.1), by some new assumptions to present the new concept of the efficiency of the proposed scheme.

Example 4.2 *Consider the following interval problem with real variables:*

(23) $$min \;\; \langle 4,2\rangle x_1 + \langle 4,3\rangle x_2 + \langle 4,1\rangle x_3 + \langle 6,5\rangle x_4 + \langle 1,9\rangle x_5 + \langle 10,1\rangle x_6,$$

s.t.

$$\langle 1,0.5\rangle x_1 + \langle 1,1\rangle x_2 + x_4 + x_6 \geq \langle 200,10\rangle,$$

$$\langle 2,1\rangle x_1 + \langle 3,2\rangle x_3 + x_5 \geq \langle 300,50\rangle,$$

$$\langle 1,1\rangle x_2 + \langle 3,2\rangle x_3 - \langle 2,1\rangle x_4 + \langle 4,3\rangle x_5 \geq \langle 500,40\rangle,$$

$$x_1, ..., x_6 \geq 0.$$

The constraints of this problem may be rewrited as bellow:

$$\begin{cases} \langle x_1 + x_2 + x_4 + x_6, 0.5x_1 + x_2 \rangle \geq \langle 200, 10 \rangle, \\ \langle 2x_1 + 3x_3 + x_5, x_1 + 2x_3 \rangle \geq \langle 500, 40 \rangle, \\ \langle x_2 + 3x_3 - 2x_4 + 4x_5, x_2 + 2x_3 - x_4 + 3x_5 \rangle \geq \langle 500, 40 \rangle, \end{cases}$$

or by utilizing our ordering (3.3) with some fixed couple of (k, l) satisfying in Proposition (3.8), we have

$$\begin{cases} (k + 0.5l)x_1 + (k + l)x_2 + kx_4 + kx_6 \geq 200k + 10l, \\ (2k + l)x_1 + (3k + 2l)x_3 + kx_5 \geq 500k + 40l, \\ (k + l)x_2 + (3k + 2l)x_3 - (2k + l)x_4 + (4k + 3l)x_5 \geq 500k + 40l, \end{cases}$$

Therefore this case is transformed to the previous case which is presented in Example (4.1) and the discussion of that status is true for this state, too.

4.3 LP with interval coefficients and variables

In the bellow, a fully interval version of linear programming is introduced. This model can handel the uncertainty concepts in framework of problem and in decision maker's solutions.

Example 4.3 *Consider the following interval problem with interval variables:*

(24)
$$min \;\; \langle 4, 2 \rangle \times \langle x_1, \xi_1 \rangle + \langle 4, 3 \rangle \times \langle x_2, \xi_2 \rangle + \langle 4, 1 \rangle \times \langle x_3, \xi_3 \rangle$$
$$+ \langle 6, 5 \rangle \times \langle x_4, \xi_4 \rangle + \langle 1, 9 \rangle \times \langle x_5, \xi_5 \rangle + \langle 10, 1 \rangle \times \langle x_6, \xi_6 \rangle,$$

s.t.

$$\langle 1, 0.5 \rangle \times \langle x_1, \xi_1 \rangle + \langle 1, 1 \rangle \times \langle x_2, \xi_2 \rangle + x_4 + x_6 \geq \langle 200, 10 \rangle,$$

$$\langle 2, 1 \rangle \times \langle x_1, \xi_1 \rangle + \langle 3, 2 \rangle \times \langle x_3, \xi_3 \rangle + x_5 \geq \langle 300, 50 \rangle,$$

$$\langle 1, 1 \rangle \times \langle x_2, \xi_2 \rangle + \langle 3, 2 \rangle \times \langle x_3, \xi_3 \rangle - \langle 2, 1 \rangle \times \langle x_4, \xi_4 \rangle + \langle 4, 3 \rangle \times \langle x_5, \xi_5 \rangle \geq \langle 500, 40 \rangle,$$

$$x_1 - \xi_1, ..., x_6 - \xi_6 \geq 0.$$

By arithmetic operators (10) and (11), this problem simplifies as follows:

(25)
$$min \;\; \langle 4x_1 + 2\xi_1 + 4x_2 + 3\xi_2 + 4x_3 + \xi_3 + 6x_4 + 5\xi_4 + x_5 + 9\xi_5 + 10x_6 + \xi_6,$$
$$2x_1 + 4\xi_1 + 3x_2 + 4\xi_2 + x_3 + 4\xi_3 + 5x_4 + 6\xi_4 + 9x_5 + \xi_5 + x_6 + 10\xi_6 \rangle,$$

s.t.

$$\begin{cases} \langle x_1 + 0.5\xi_1 + x_2 + \xi_2 + x_4 + x_6, 0.5x_1 + \xi_1 + x_2 + \xi_2 \rangle \geq \langle 200, 10 \rangle \\ \langle 2x_1 + \xi_1 + 3x_3 + 2\xi_3 + x_5, x_1 + 2\xi_1 + 2x_3 + 3\xi_3 \rangle \geq \langle 300, 50 \rangle, \\ \langle x_2 + \xi_2 + 3x_3 + 2\xi_3 - 2x_4 - \xi_4 + 4x_5 + 3\xi_5, \\ \qquad x_2 + \xi_2 + 2x_3 + 3\xi_3 - x_4 - 2\xi_4 + 3x_5 + 4\xi_5 \rangle \geq \langle 500, 40 \rangle, \end{cases}$$

The objective function can be rewrite as bellow:

$$min \;\; \langle 4, 2 \rangle x_1 + \langle 4, 3 \rangle x_2 + \langle 4, 1 \rangle x_3 + \langle 6, 5 \rangle x_4 + \langle 1, 9 \rangle x_5 + \langle 10, 1 \rangle x_6,$$

$$+\langle 2,4\rangle\xi_1+\langle 3,4\rangle\xi_2+\langle 1,4\rangle\xi_3+\langle 5,6\rangle\xi_4+\langle 9,1\rangle\xi_5+\langle 1,10\rangle\xi_6.$$

Also respect to the ordering (3.3) with some fixed couple of (k,l) fulfilling in Proposition (3.8), we can transform the constraint as follows:

$$\begin{cases} (k+0.5l)x_1+(0.5k+l)\xi_1+(k+l)x_2+(k+l)\xi_2+kx_4+kx_6\geq 200k+10l \\ (2k+l)x_1+(k+2l)\xi_1+(3k+2l)x_3+(2k+3l)\xi_3+kx_5\geq 300k+50l, \\ (k+l)x_2+(k+l)\xi_2+(3k+2l)x_3+(2k+3l)\xi_3-(2k+l)x_4-(k+2l)\xi_4 \\ +(4k+3l)x_5+(3k+4l)\xi_5,\geq 500k+40l. \end{cases}$$

Thus a problem with interval objective function and real restriction is obtained and the result of Example (4.1) is true for this state, too. Only note that this problem has twice as variables as case 1 has.

5 Conclusion and future direction

It is reasonable in each area of mathematics that mathematician want to work on an appropriate algebraic structure such as group or field. We showed our known intervals can not create a field by classic definition of operators. Therefore we define addition and multiplication in direct form and obtain two formulas for subtraction and division. In addition, we interpret the operators with level of uncertainty. These new operators may be produce some interval with negative width which can be appear in solution of fuzzy systems. We inserted these new intervals to the set of classic intervals and showed that this new set is a field. Then by employing non-algebraic numbers, we extend a total order on a sufficiently large subset of interval field. Then we solve three version of interval linear programming and compare our results respect to that ordering with some previous nice orders. These examples demonstrate the efficiency of our schemes. Introducing on the duality concepts of interval linear programming based on the mentioned preliminaries is left to the next works. The method also may be extended into multi criteria optimization analogues to [30].

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